British Mathematical Olympiad

Round 1 : Wednesday 19th January 1994

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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1. Starting with any three digit number n (such as $n = 625$) we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs, and the product of all three digits.
   (i) Find the value of $n/f(n)$ when $n = 625$. (The answer is an integer!)
   (ii) Find all three digit numbers such that the ratio $n/f(n)$=1.

2. In triangle ABC the point X lies on BC.
   (i) Suppose that $\angle BAC = 90°$, that X is the midpoint of BC, and that $\angle BAX$ is one third of $\angle BAC$. What can you say (and prove!) about triangle ACX?
   (ii) Suppose that $\angle BAC = 60°$, that X lies one third of the way from B to C, and that AX bisects $\angle BAC$. What can you say (and prove!) about triangle ACX?

3. The sequence of integers $u_0, u_1, u_2, u_3, \ldots$ satisfies $u_0 = 1$ and
   $$u_{n+1}u_{n-1} = ku_n \quad \text{for each} \quad n \geq 1,$$
   where k is some fixed positive integer. If $u_{2000} = 2000$, determine all possible values of k.

4. The points Q, R lie on the circle γ, and P is a point such that PQ, PR are tangents to γ. A is a point on the extension of PQ, and γ' is the circumcircle of triangle PAR. The circle γ' cuts γ again at B, and AR cuts γ at the point C. Prove that $\angle PAR = \angle ABC$.

5. An *increasing* sequence of integers is said to be **alternating** if it *starts* with an *odd* term, the second term is even, the third term is odd, the fourth is even, and so on. The empty sequence (with no term at all!) is considered to be alternating. Let $A(n)$ denote the number of alternating sequences which only involve integers from the set $\{1, 2, \ldots, n\}$. Show that $A(1) = 2$ and $A(2) = 3$. Find the value of $A(20)$, and prove that your value is correct.